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LETTER TO THE EDITOR

Path integral formalism for $SU_q(2)$ coherent states

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Abstract. A path integral formulation in the representation of coherent states for the quantum group $SU_q(2)$ is introduced. An expression for the transition amplitude connecting two $SU_q(2)$ coherent states is constructed, and the corresponding canonical equations of motion are also derived.

Both path integral [1] and coherent states [2, 3] have played major roles in the study of quantum mechanical systems, particularly for establishing the correspondence between classical and quantum physics. The use of coherent states to provide an alternative method of obtaining the phase space path integral and Hamilton's equations of motion was pioneered by Klauder [4]. This technique has recently been extended to include a formulation in terms of generalized coherent states for SU(1, 1) [5], SU(2) [6] and Osp(1/2, R) [7]. More recently, quantum algebras or quantum groups, which may be viewed as deformations of classical Lie algebras or groups, have begun to be investigated [8, 9], and the coherent states for the quantum group $SU_q(2)$ are also defined [10, 11]. Furthermore, in [12, 13], the mathematical properties of these q-coherent states, the supercoherent states, path integrals and their semiclassical limit are discussed. In this letter, we shall extend the previous path integral formalism of coherent states to the quantum group $SU_q(2)$. We introduce the normalized $SU_q(2)$ coherent states instead of the unnormalized ones in [11], and construct an invariant measure of integration for the normalized coherent states. Later, we present the path integral formulation of the transition amplitude between two $SU_q(2)$ coherent states and derive the classical equations of motion for the system.

Let us consider the quantum group $SU_q(2)$, which is generated algebraically by the operators $\hat{J}_+$, $\hat{J}_-$ and $\hat{J}_0$ obeying the relations

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm \qquad [\hat{J}_+, \hat{J}_-] = \frac{q^{\hat{J}_0} - q^{-\hat{J}_0}}{q^{1/2} - q^{-1/2}} = [2\hat{J}_0] \tag{1}$$

where q is a real number. The $SU_q(2)$ unitary irreps [8] are characterized by j, which may take the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and their carrier space is spanned by $2j+1$ vectors $\{|j, m\rangle, m = -j, -j+1, \ldots, j\}$ such that

$$\begin{aligned} \hat{J}_0|j, m\rangle &= m|j, m\rangle \\ \hat{J}_\pm|j, m\rangle &= ([j \mp m][j \pm m + 1])^{1/2}|j, m \pm 1\rangle. \end{aligned} \tag{2}$$

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The $SU_q(2)$ coherent states $|z\rangle$ for a given irrep j can be defined in terms of a q-exponential as

$$|z\rangle = N_j(\bar{z}z;\underline{q})e_{\underline{q}}(z\hat{J}_+)|j,-j\rangle$$
$$= N_j(\bar{z}z;\underline{q}) \sum_{m=-j}^{j} \left(\frac{[2j]!}{[j+m]![j-m]!}\right)^{1/2} z^{j+m}|j,m\rangle. \qquad (3)$$

Hereafter the overbar stands for the complex conjugation. The normalization constant $N_j(\bar{z}z;\underline{q})$ is given by

$$N_j^{-2}(\bar{z}z;\underline{q}) = \sum_{m=-j}^{j} \frac{[2j]!}{[j+m]![j-m]!}(\bar{z}z)^{j+m}. \qquad (4)$$

Furthermore, the scalar product of $SU_q(2)$ coherent states may be expressed as

$$\langle z_1|z_2\rangle = N_j(\bar{z}_1z_1;\underline{q})N_j(\bar{z}_2z_2;\underline{q})N_j^{-2}(\bar{z}_1z_2;\underline{q}). \qquad (5)$$

There exists another form for the normalization constant $N_j(\bar{z}z;\underline{q})$:

$$N_j^{-2}(\bar{z}z;\underline{q}) = \prod_{k=1}^{2j}(1+\bar{z}z\underline{q}^{k-j-1/2}). \qquad (6)$$

It is easy to verify that formulae (4) and (6) are equivalent by means of mathematical induction. The decomposition of unity for $SU_q(2)$ coherent states and the measure of integration, which differs from that of [11] because the coherent states are normalized to unity, can be written as

$$I = \int |z\rangle\langle z|\, \mathrm{d}\sigma(z) \qquad (7)$$

$$\mathrm{d}\sigma(z) = \frac{[2j+1]}{2\pi}(1+\bar{z}z\underline{q}^{-j-1/2})^{-1}(1+\bar{z}z\underline{q}^{j+1/2})^{-1}\,\mathrm{d}_{\underline{q}}|z|^2\,\mathrm{d}\theta. \qquad (8)$$

Note that the normal integration $\mathrm{d}\theta$ goes from 0 to 2π, while the q-integration $\mathrm{d}_q|z|^2$ goes from 0 to ∞. In the derivation, use is made of a q-analogue of the beta integration, which, for $x, y \in \mathbb{N}$, can be written as [11]

$$\int_0^\infty t^{x-1}\prod_{k=0}^{x+y-1}(1-at\underline{q}^{k-1/2})^{-1}\,\mathrm{d}_{\underline{q}}t = (-a)^{-x}\underline{q}^{-x(x+y-2)/2}\frac{[x-1]![y-1]!}{[x+y-1]!}. \qquad (9)$$

Let us consider a Hamiltonian $\hat{H}$, which is constructed from the infinitesimal operators $\hat{J}_\pm$ and $\hat{J}_0$ of $SU_q(2)$. Since $\hat{H}$ conserves the quantum number j, we hereafter consider only the states with fixed j. The transition amplitude from the coherent state $|z_0\rangle$ at time t_0 to the coherent state $|z'\rangle$ at time t' is given by

$$T(z't';z_0t_0) = \left\langle z'\left|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t'-t_0)\right)\right|z_0\right\rangle. \qquad (10)$$

As usual, we divide the time interval $\Delta T = t'-t_0$ into n equal parts $\varepsilon = \Delta T/n$ and take the limit $n\to\infty$

$$T = \lim_{n\to\infty}\left\langle z'\left|\left(1-\frac{\mathrm{i}}{\hbar}\hat{H}\varepsilon\right)^n\right|z_0\right\rangle. \qquad (11)$$

Inserting the decomposition of unity of (7) into each time interval of (11) leads to the expression

$$T=\lim_{n\to\infty}\int\ldots\int\prod_{k=1}^{n-1}\mathrm{d}\sigma(z_k)\prod_{k=1}^{n}\left\langle z_k\left|\left(1-\frac{\mathrm{i}}{\hbar}\hat{H}\varepsilon\right)\right|z_{k-1}\right\rangle$$

$$=\lim_{n\to\infty}\int\ldots\int\prod_{k=1}^{n-1}\mathrm{d}\sigma(z_k)\prod_{k=1}^{n}\langle z_k|z_{k-1}\rangle\prod_{k=1}^{n}\left\{1-\frac{\mathrm{i}\varepsilon}{\hbar}\frac{\langle z_k|\hat{H}|z_{k-1}\rangle}{\langle z_k|z_{k-1}\rangle}\right\} \tag{12}$$

where $z_n=z'$. The term in the curly bracket in (12) is the simplest to handle, and to first order in ε it can be replaced by the exponential of the expectation value of the Hamiltonian. Next, by using the identity $\langle z_k|z_{k-1}\rangle=\exp(\ln\langle z_k|z_{k-1}\rangle)$ and the explicit form of the scalar product (i.e. equations (5) and (6)) the factor for the scalar products of the coherent states is expressed as

$$\prod_{k=1}^{n}\langle z_k|z_{k-1}\rangle=\exp\left\{\sum_{k=1}^{n}\varepsilon\frac{1}{\varepsilon}\ln\langle z_k|z_{k-1}\rangle\right\}$$

$$=\exp\sum_{k=1}^{n}\varepsilon\left\{\tfrac{1}{2}\sum_{l=1}^{2j}\frac{\underline{q}^{l-j-1/2}}{1+\bar{z}_kz_k\underline{q}^{l-j-1/2}}\left(z_k\frac{\Delta\bar{z}_k}{\varepsilon}-\bar{z}_k\frac{\Delta z_k}{\varepsilon}\right)+\frac{1}{\varepsilon}\mathcal{O}((\Delta z_k)^2)\right\}$$

$$\to\exp\int\mathrm{d}t\left\{\tfrac{1}{2}\sum_{l=1}^{2j}\frac{\underline{q}^{l-j-1/2}}{1+\bar{z}z\underline{q}^{l-j-1/2}}(z\dot{\bar{z}}-\dot{z}\bar{z})\right\} \tag{13}$$

where the dot denotes the time derivative and $\Delta z_k=z_k-z_{k-1}$.

The transition amplitude can then be written in the following formal manner:

$$\begin{aligned}T&=\int\mathscr{D}\sigma[z(t)]\exp\left(\frac{\mathrm{i}}{\hbar}S\right)\\ S&=\int_{t_0}^{t'}\mathscr{L}(z(t),\dot{z}(t),\bar{z}(t),\dot{\bar{z}}(t))\,\mathrm{d}t\end{aligned} \tag{14}$$

where S is the action and the 'Lagrangian' $\mathscr{L}$ is given by

$$\begin{aligned}\mathscr{L}&=\frac{\mathrm{i}\hbar}{2}\sum_{l=1}^{2j}\frac{\underline{q}^{l-j-1/2}}{1+\bar{z}(t)z(t)\underline{q}^{l-j-1/2}}\{\bar{z}(t)\dot{z}(t)-\dot{\bar{z}}(t)z(t)\}-\mathscr{H}\\ \mathscr{H}&=\langle z|\hat{H}|z\rangle\end{aligned} \tag{15}$$

which can be rewritten as

$$\mathscr{L}=\left\langle z(t)\left|\left(\mathrm{i}\hbar\frac{\partial}{\partial t}-\hat{H}\right)\right|z(t)\right\rangle \tag{16}$$

with the aid of

$$\left\langle z\left|\frac{\partial}{\partial t}\right|z\right\rangle=\lim_{z_1\to z}\left\langle z_1\left|\frac{\partial}{\partial t}\right|z\right\rangle$$

$$=\lim_{z_1\to z}\left(\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\{N_j(\bar{z}_1z_1;\underline{q})N_j(\bar{z}z;\underline{q})N_j^{-2}(\bar{z}_1z;\underline{q})\}. \tag{17}$$

To arrive at the classical limit, we consider the case $S \gg \hbar$. The main contribution to the transition amplitude T then comes from the path which makes the action stationary with fixed endpoint conditions $z_0 = z(t_0)$, $z' = z(t')$:

$$0 = \delta S = \int_{t_0}^{t'} \left(\frac{\partial \mathscr{L}}{\partial z} \delta z + \frac{\partial \mathscr{L}}{\partial \dot{z}} \delta \dot{z} + \mathrm{cc} \right) \mathrm{d}t$$
$$= \int_{t_0}^{t'} \left\{ \left[\frac{\partial \mathscr{L}}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathscr{L}}{\partial \dot{z}} \right) \right] \delta z + \mathrm{cc} \right\} \mathrm{d}t. \tag{18}$$

As the variations δz and $\delta \bar{z}$ are independent and arbitrary, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathscr{L}}{\partial \dot{z}} \right) - \frac{\partial \mathscr{L}}{\partial z} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathscr{L}}{\partial \dot{\bar{z}}} \right) - \frac{\partial \mathscr{L}}{\partial \bar{z}} = 0. \tag{19}$$

Using the expression (15) for $\mathscr{L}$, we cast equation (19) into the 'canonical' form as

$$\dot{z} = -\frac{\mathrm{i}}{\hbar} \left\{ \sum_{l=1}^{2j} \underline{q}^{l-j-1/2} (1 + \bar{z} z \underline{q}^{l-j-1/2})^{-2} \right\}^{-1} \frac{\partial \mathscr{H}}{\partial \bar{z}}$$
$$\dot{\bar{z}} = \frac{\mathrm{i}}{\hbar} \left\{ \sum_{l=1}^{2j} \underline{q}^{l-j-1/2} (1 + \bar{z} z \underline{q}^{l-j-1/2})^{-2} \right\}^{-1} \frac{\partial \mathscr{H}}{\partial z}. \tag{20}$$

Now we define the Poisson bracket by

$$\{\mathrm{A}, \mathrm{B}\} = \frac{\mathrm{i}}{\hbar} \left\{ \sum_{l=1}^{2j} \underline{q}^{l-j-1/2} (1 + \bar{z} z \underline{q}^{l-j-1/2})^{-2} \right\}^{-1} \left(\frac{\partial \mathrm{A}}{\partial \bar{z}} \frac{\partial \mathrm{B}}{\partial z} - \frac{\partial \mathrm{A}}{\partial z} \frac{\partial \mathrm{B}}{\partial \bar{z}} \right) \tag{21}$$

for arbitrary functions A and B of z and $\bar{z}$. As is easily verified, this bracket satisfies the antisymmetricity and Jacobi's identity. Then equation (20) is brought into the form

$$\dot{z} = \{z, \mathscr{H}\} \qquad \dot{\bar{z}} = \{\bar{z}, \mathscr{H}\}. \tag{22}$$

Alternatively, using the angle variables (θ, φ) through stereographic projection

$$z = \tan\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\varphi} \qquad (0 \leqslant \theta \leqslant \pi, 0 \leqslant \varphi < 2\pi) \tag{23}$$

we can rewrite equation (20) as

$$\dot{\theta} = \{\theta, \mathscr{H}\} \qquad \dot{\varphi} = \{\varphi, \mathscr{H}\} \tag{24}$$

where the Poisson bracket is given by

$$\{\mathrm{A}, \mathrm{B}\} = \frac{1}{\hbar} \cot \frac{\theta}{2} \cos^2 \frac{\theta}{2} \left\{ \sum_{l=1}^{2j} \underline{q}^{l-j-1/2} \left(1 + \tan^2 \frac{\theta}{2} \underline{q}^{l-j-1/2} \right)^{-2} \right\}^{-1} \left(\frac{\partial \mathrm{A}}{\partial \varphi} \frac{\partial \mathrm{B}}{\partial \theta} - \frac{\partial \mathrm{A}}{\partial \theta} \frac{\partial \mathrm{B}}{\partial \varphi} \right). \tag{25}$$

Finally, we wish to point out that all the results in this letter are identical with those in [6], i.e. in the case of SU(2), as $q \to 1$. This is just what we want.

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